Introduction to Gelfand Theory

Junekey Jeon

ETRI

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- Some of those spaces are **not just vector spaces**
  - They are ***algebras***
- Studying algebras is radically different from studying vector spaces
  - *Rings* vs *Abelian groups*

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   - $M(G)$, algebra of complex regular Borel measures on G

Applications

- Spectral theorem and functional calculus
- Operator semigroup theory and its applications to PDE
- Abstract harmonic analysis
- Ergodic theory
- Quantum physics
- And more...

Definition

Definition 2.1 (Banach algebra)

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  - Approximate identity: a net $(e_\alpha)_{\alpha \in D}$ such that $\lim_{\alpha \in D} x e_\alpha = \lim_{\alpha \in D} e_\alpha x = x$ for all $x \in A$

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- Algebraic structure and topological structure interoperate tightly!

Spectrum and Invertibility

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Proposition 2.6

$A^\times$ is an open set.

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$$\sigma(a) := \left\{\lambda \in \mathbb{C} : a - \lambda 1 \notin A^\times\right\}$$

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- $\sigma(a)$ encodes many useful information about a

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  - *(Spectral radius formula)*

$$|a|_\sigma := \sup \{|\lambda| : \lambda \in \sigma(a)\} = \lim_{n \to \infty} \|a^n\|^{1/n}$$

(by Cauchy's integral formula)

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Definition 3.3 (Spectrum)

Let us denote the set of all nontrivial homomorphisms $\varphi : A \to \mathbb{C}$ as $\mathrm{Spec} A$ and call it the *spectrum* of A.

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  - $\mathrm{Spec} A$ is a good candidate!

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- The Gelfand transform is injective iff A is semisimple

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  - The above bijection is in fact a homeomorphism
  - The Gelfand transform $\hat{\ } : \ell^1(\mathbb{Z}) \to \mathcal{C}(S^1)$ is the *Fourier transform* on $\mathbb{Z}$

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Corollary 3.7

The spectrum $\sigma(a)$ is precisely the range of the function $\hat{a} : \mathrm{Spec} A \to \mathbb{C}$. In particular, $|a|_\sigma = \|\hat{a}\|_\infty$.

Stone-Weierstrass Theorem

Theorem 4.1 (Stone-Weierstrass Theorem)

Let X be a compact Hausdorff space. Then a subalgebra A of $\mathcal{C}(X)$ is uniformly dense in $\mathcal{C}(X)$ if the following conditions hold:

1. *A is unital; that is, A contains the constant function 1,*
2. *A is a $*$-subalgebra; that is, A is closed under the pointwise complex conjugation, and*
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  - separates points in $\mathrm{Spec} A$, by definition of $\mathrm{Spec} A$

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- Most of function algebras w/ uniform norm are C^*-algebras

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A Banach algebra A endowed with an involution $* : A \to A$ is called a *C^*-algebra*, if:

1. $*$ is antilinear; that is, $(\alpha x + \beta y)^* = \bar{\alpha} x^* + \bar{\beta} y^*$ for all $\alpha, \beta \in \mathbb{C}$, $x, y \in A$.
2. $(xy)^* = y^* x^*$ for all $x, y \in A$.
3. $\|x^* x\| = \|x\|^2$ for all $x \in A$.

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- Most of function algebras w/ uniform norm are C^*-algebras
- The algebra $L(E)$ of all bounded linear operators on a Hilbert space E is a C^*-algebra
- The algebra $L^1(G)$ nor $M(G)$ are not C^*-algebras in general

Self-adjoint and Normal Elements

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For the general case,

$$\|x\|^2 = \|x^*x\| = |x^*x|_\sigma \overset{(a)}{\leq} |x|_\sigma^2 \leq \|x\|^2$$

where (a) follows from the spectral radius formula. □

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Suffices to show that self-adjoint elements become real numbers.
(1) Estimate using $\exp$.
(2) Estimate using square. □

Commutative Gelfand-Naimark Theorem

Corollary 4.7 (Gelfand-Naimark)

For a commutative unital C^*-algebra A, the Gelfand transform $\hat{\ } : A \to \mathcal{C}(\mathrm{Spec} A)$ is an isometric $*$-isomorphism.

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- $\mathrm{Spec}(\,\cdot\,)$ and $\mathcal{C}(\,\cdot\,)$ are adjoint pairs
  - (category of commutative unital C^*-algebras)$\cong$ (category of compact Hausdorff spaces)

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- Suppose A is a C^*-algebra and $a \in A$

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Injective: if $\varphi(a) = \psi(a)$, then $\varphi = \psi$.
Surjective: the spectrum taken inside B is exactly $\sigma(a)$.
(Nontrivial) □

Functional Calculus

Corollary 5.2 (Functional calculus theorem)

When a is normal, there is a natural isometric $*$-isomorphism $B \cong \mathcal{C}(\sigma(a))$, given by $a \mapsto (\lambda \mapsto \lambda)$. Under this isomorphism, we have $\sigma(f(a)) = f[\sigma(a)]$ for each $f \in \mathcal{C}(\sigma(a))$.

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- The polynomial $p(\lambda) = c_n\lambda^n + \cdots + c_0$ corresponds to the element $p(a) = c_n a^n + \cdots + c_0$ in A
- By Stone-Weierstrass theorem, $\mathcal{C}(\sigma(a))$ is the set of functions on $\sigma(a)$ that can be uniformly approximated by polynomials in λ and $\overline{\lambda}$.

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- By Riesz representation theorem, such a functional can be represented as a complex regular Borel measure $\rho_{u,v}$ on $\sigma(T)$
- Define $\rho : \mathcal{B}(\sigma(T)) \to L(E)$ as $\langle u, \rho(G)v \rangle := \rho_{u,v}(G)$, then

$$\langle u, f(T)v \rangle = \int_{\sigma(T)} f(\lambda) \, \langle u, d\rho(\lambda)v \rangle$$

for each $f \in \mathcal{C}(\sigma(T))$

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$$\rho_{u,v}(K) = \lim_{f \downarrow \mathbb{1}_K} \int_{\sigma(A)} f(\lambda) d\rho_{u,v}(\lambda) = \lim_{f \downarrow \mathbb{1}_K} \langle u, f(T)v \rangle ,$$

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$$\left\langle u, \rho(K)^2 v\right\rangle = \lim_{f\downarrow \mathbb{1}_K} \langle u, \rho(K) f(T)v\rangle = \lim_{f\downarrow \mathbb{1}_K} \lim_{g\downarrow \mathbb{1}_K} \langle u, g(T) f(T)v\rangle$$

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Theorem 5.3 (Spectral theorem for bounded normal operators)

If T is a bounded normal operator on a Hilbert space E, then there uniquely exists a projection-valued regular Borel measure ρ_T on $\sigma(T)$ such that
$$\langle u, f(T)v\rangle = \int_{\sigma(T)} f(\lambda)\,\langle u, d\rho_T(\lambda)v\rangle$$
for all $f \in \mathcal{C}(\sigma(T))$ and $u, v \in E$.