Conic Intrinsic Volumes and Conic Linear Programming

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UCSD

- (2015,*D. Amelunxen and P. Bürgisser*) Intrinsic volumes of symmetric cones and applications in convex programming
- (2018 arXiv version) Intrinsic volumes of symmetric cones

$$
\begin{aligned}
\text{maximize} \quad & \langle z, x \rangle \\
\text{subject to} \quad & \langle a_i, x \rangle = b_i, \quad i = 1, \cdots, m, \\
& x \in C
\end{aligned}
\tag{CP}
$$

- $a_1, \cdots, a_m, z \in \mathbb{R}^d$, $b_1, \cdots, b_m \in \mathbb{R}$
- C is a closed convex cone in $\mathbb{R}^d$

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Theorem 1

When $a_1, \cdots, a_m, z$ and $b_1, \cdots, b_m$ are iid standard normal,

$$\Pr[\text{(CP) is infeasible}] = \sum_{j=0}^{m-1} V_j(C),$$

$$\Pr[\text{(CP) is unbounded}] = \sum_{j=m+1}^{d} V_j(C).$$

Furthermore, for a conic Borel set $M \subseteq \mathbb{R}^d$,

$$\Pr[\text{sol(CP)} \in M] = \Phi_m(C, M).$$

- $\Phi_j(C, \cdot\,)$ is called the jth **(conic) curvature measure** of C
- $V_j(C) := \Phi_j(C, \mathbb{R}^d)$ is called the jth **(conic) intrinsic volume** of C

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- In particular, if $A_1, \cdots, A_m, Z \sim \mathrm{GOE}(d)$ or $\mathrm{GUE}(d)$ or $\mathrm{GSE}(d)$, $b_1, \cdots, b_m$ are iid standard normal from $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$, then for semidefinite program

$$\begin{aligned} \text{maximize} \quad & \mathrm{tr}(Z^\dagger X) \\ \text{subject to} \quad & \mathrm{tr}(A_i^\dagger X) = b_i, \quad i = 1, \cdots, m, \\ & X \succeq 0 \end{aligned} \tag{SDP}$$

we can compute

$$\Pr[(\mathrm{SDP}) \text{ is infeasible}], \ \Pr[(\mathrm{SDP}) \text{ is unbounded}],$$

and also

$$\Pr[\mathrm{rank}(\mathrm{sol}(\mathrm{CP})) = r]$$

for each $0 \leq r \leq d$, by obtaining an explicit formula for the intrinsic volumes and the curvature measures of the positive-semidefinite cone evaluated at the set of rank r matrices

β \ j	1	2	3	4	5	6	7	8	9
1	$\frac{\sqrt{2}}{4} - \frac{1}{4}$	$\frac{\sqrt{2}}{2\pi}$	$\frac{1}{2} - \frac{\sqrt{2}}{4}$	0	0	0	0	0	0
2	$\frac{3}{16} - \frac{1}{2\pi}$	$\frac{1}{4\pi}$	$\frac{1}{2\pi}$	$\frac{1}{2\pi}$	$\frac{3}{16} - \frac{3}{8\pi}$	0	0	0	0
4	$\frac{11}{64} - \frac{8}{15\pi}$	$\frac{1}{40\pi}$	$\frac{4}{15\pi} - \frac{1}{16}$	$\frac{19}{120\pi}$	$\frac{3}{32}$	$\frac{2}{5\pi}$	$\frac{1}{16} + \frac{1}{6\pi}$	$\frac{1}{5\pi}$	$\frac{7}{64} - \frac{7}{24\pi}$

Table 1: The values of $\Phi_j(\beta, 3, 1)$.

	V_0	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8
$\mathcal{C}_{\beta,1}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	0	0	0
$\mathcal{C}_{1,2}$	$\frac{1}{2} - \frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{4}$	$\frac{1}{2} - \frac{\sqrt{2}}{4}$	0	0	0	0	0
$\mathcal{C}_{2,2}$	$\frac{1}{4} - \frac{1}{2\pi}$	$\frac{1}{4}$	$\frac{1}{\pi}$	$\frac{1}{4}$	$\frac{1}{4} - \frac{1}{2\pi}$	0	0	0	0
$\mathcal{C}_{4,2}$	$\frac{1}{4} - \frac{2}{3\pi}$	$\frac{1}{8}$	$\frac{2}{3\pi}$	$\frac{1}{4}$	$\frac{2}{3\pi}$	$\frac{1}{8}$	$\frac{1}{4} - \frac{2}{3\pi}$	0	0
$\mathcal{C}_{1,3}$	$\frac{1}{4} - \frac{\sqrt{2}}{2\pi}$	$\frac{\sqrt{2}}{4} - \frac{1}{4}$	$\frac{\sqrt{2}}{2\pi}$	$1 - \frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2\pi}$	$\frac{\sqrt{2}}{4} - \frac{1}{4}$	$\frac{1}{4} - \frac{\sqrt{2}}{2\pi}$	0	0
$\mathcal{C}_{2,3}$	$\frac{1}{8} - \frac{3}{8\pi}$	$\frac{3}{16} - \frac{1}{2\pi}$	$\frac{1}{4\pi}$	$\frac{1}{2\pi}$	$\frac{3}{16} + \frac{1}{8\pi}$	$\frac{3}{16} + \frac{1}{8\pi}$	$\frac{1}{2\pi}$	$\frac{1}{4\pi}$	...
$\mathcal{C}_{4,3}$	$\frac{1}{8} - \frac{47}{120\pi}$	$\frac{11}{64} - \frac{8}{15\pi}$	$\frac{1}{40\pi}$	$\frac{4}{15\pi} - \frac{1}{16}$	$\frac{19}{120\pi}$	$\frac{3}{32}$	$\frac{13}{120\pi} + \frac{7}{64}$	$\frac{11}{30\pi} + \frac{1}{16}$	...

Table 2: Intrinsic volumes of $\mathcal{C}_{\beta,n}$ for $n = 1, 2, 3$ (the missing entries for $\mathcal{C}_{2,3}$ and $\mathcal{C}_{4,3}$ are obtained via $V_j(\mathcal{C}_{\beta,n}) = V_{d_{\beta,n}-j}(\mathcal{C}_{\beta,n})$).

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- $\Phi_j(C, M)$: some sort of "localization" of $V_j(C)$ into M

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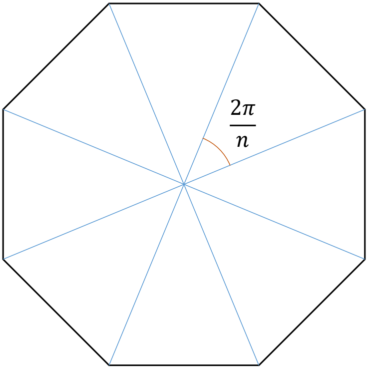
- Let us look at *Euclidean* intrinsic volumes first!

- Regular n-gon inscribed in the circle of radius r in 2D
  - Number of vertices: n
  - Circumference: $2nr\sin\frac{\pi}{n}$
  - Area: $\frac{nr^2}{2}\sin\frac{2\pi}{n}$



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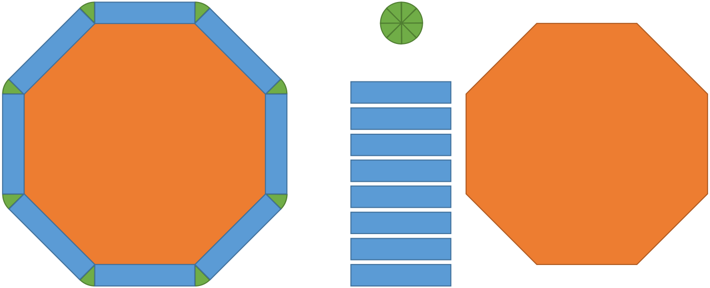
- Consider the area of the ϵ-neighborhood
  - Green: $\pi\epsilon^2$
  - Blue: $2nr\epsilon\sin\frac{\pi}{n}$
  - Red: $\frac{nr^2}{2}\sin\frac{2\pi}{n}$



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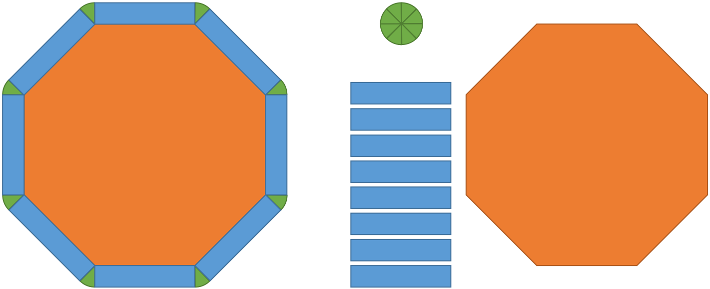
- Normalize by the volume of $(d-j)$-dimensional ball:
  - $V_0 = \pi\epsilon^2/\pi\epsilon^2 = 1$
  - $V_1 = \left(2nr\epsilon \sin \frac{\pi}{n}\right)/2\epsilon = nr \sin \frac{\pi}{n}$
  - $V_2 = \left(\frac{nr^2}{2} \sin \frac{2\pi}{n}\right)/1 = \frac{nr^2}{2} \sin \frac{2\pi}{n}$



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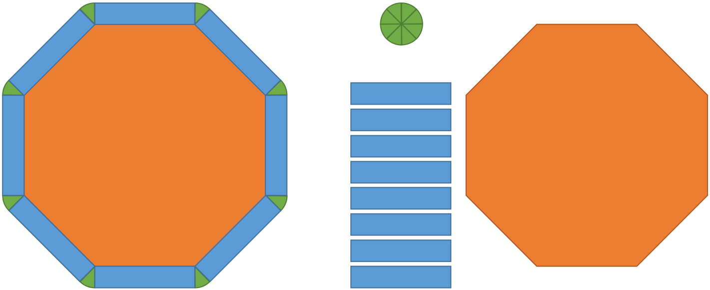
- The area of the ϵ-neighborhood
  $= V_0 \operatorname{vol}_2(B_\epsilon^2) + V_1 \operatorname{vol}_1(B_\epsilon^1) + V_2 \operatorname{vol}_0(B_\epsilon^0)$



- In general, for a compact convex set K in $\mathbb{R}^d$, there uniquely exists $V_0(K), \cdots, V_d(K) \geq 0$, called the **intrinsic volumes** of K, such that

$$\mathrm{vol}_d(K + B_\epsilon^d) = \sum_{j=0}^{d} V_j(K)\,\mathrm{vol}_{d-j}(B_\epsilon^{d-j})$$

- The above equation is called the *Steiner's formula*

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- V_j is continuous with respect to the Hausdorff metric
  - E.g., $\mathrm{vol}_2(B_r^2 + B_\epsilon^2) = \pi(r+\epsilon)^2 = 1 \cdot \pi\epsilon^2 + \pi r \cdot 2\epsilon + \pi r^2 \cdot 1$
    - $V_0(B_r^2) = 1 = \lim_{n\to\infty} 1$
    - $V_1(B_r^2) = \pi r = \lim_{n\to\infty} nr \sin\frac{\pi}{n}$
    - $V_2(B_r^2) = \pi r^2 = \lim_{n\to\infty} \frac{nr^2}{2} \sin\frac{2\pi}{n}$

- For a closed convex cone $C \subseteq \mathbb{R}^d$, we can define the **conic intrinsic volumes** $V_0(C), \cdots, V_d(C)$ of C using a spherical analogue of the Steiner's formula

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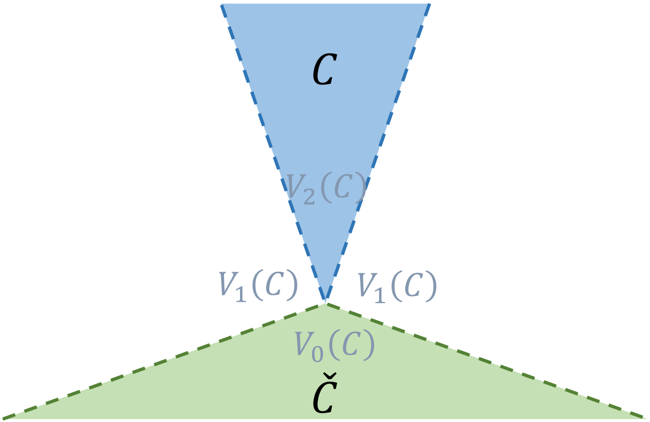
- For a *polyhedral* convex cone C, one can show

$$V_j(C) = \sum_{F \in \mathcal{F}_j} \Pr_{x \sim \mathcal{N}(0, I_d)} [\Pi_C(x) \in F]$$

where $\mathcal{F}_j$ is the set of j-dimensional (open) faces of C

- When C is a 2-dimensional cone with the central angle θ,
  - $V_2(C) = \frac{\theta}{2\pi}$
  - $V_1(C) = \frac{1}{2}$
  - $V_0(C) = \frac{\pi-\theta}{2\pi}$



- More generally, for a closed convex cone $C \subseteq \mathbb{R}^d$, we can define the **conic curvature measures** $\Phi_0(C, \cdot), \cdots, \Phi_d(C, \cdot)$ of C using a generalized Steiner's formula

- For a *polyhedral* convex cone C, one can show

$$\Phi_j(C, M) = \sum_{F \in \mathcal{F}_j} \Pr_{x \sim \mathcal{N}(0, I_d)} [\Pi_C(x) \in F \cap M]$$

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- When C is a 2-dimensional cone,
  - $\Phi_2(C, M) = (\text{central angle of } C \cap M)/2\pi$
  - $\Phi_1(C, M) = (\# \text{ of boundary edges of } C \text{ contained in } M)/2$
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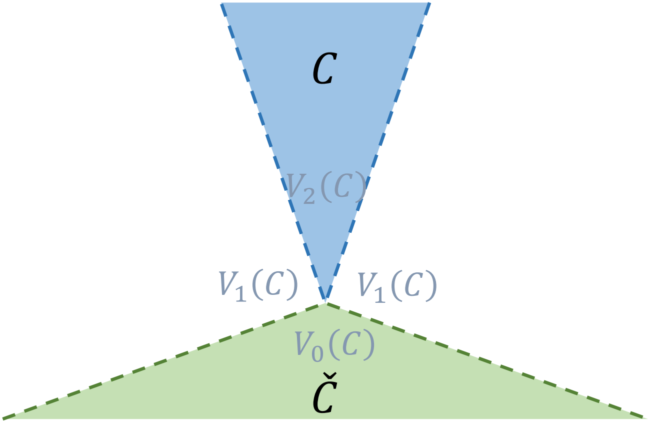

Lemma 2

Let $v, w \in S^{d-1}$, $\langle v, w \rangle = 0$, *and* $L := \operatorname{span}\{v, w\}$. *Then for a closed convex cone* $C \subseteq \mathbb{R}^d$, *we have*

$$\begin{aligned}
&\sup\{\langle v, x \rangle : x \in C,\ \langle w, x \rangle = 1\} \\
&\qquad = \sup\{\langle v, x \rangle : x \in \Pi_L(C),\ \langle w, x \rangle = 1\}, \\
&\operatorname{Argmax}\{\langle v, x \rangle : x \in C,\ \langle w, x \rangle = 1\} \\
&\qquad = C \cap \Pi_L^{-1}\left[\operatorname{Argmax}\{\langle v, x \rangle : x \in \Pi_L(C),\ \langle w, x \rangle = 1\}\right].
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\end{aligned}$$

Proof.

$x = x_L + x_L^{\perp} \Rightarrow \langle v, x\rangle = \langle v, x_L\rangle$, $\langle w, x\rangle = \langle w, x_L\rangle$, so

$$\begin{aligned}
&\sup\{\langle v, x\rangle : x \in C,\ \langle w, x\rangle = 1\} \\
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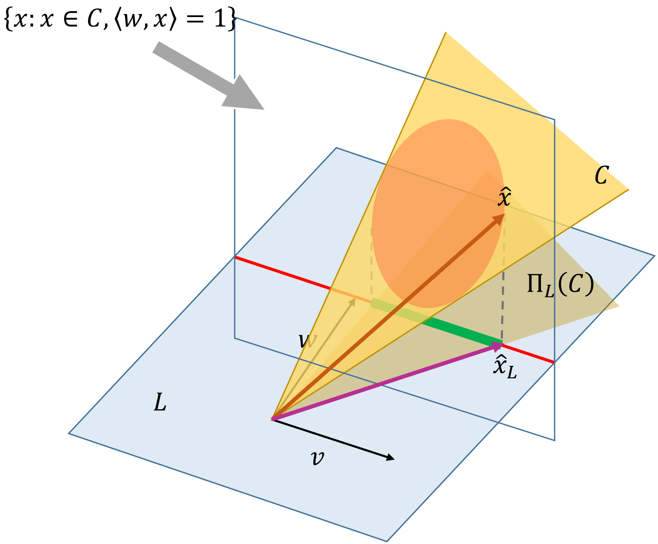
\end{aligned}$$

and similarly the second claim follows. □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- $W \coloneqq \left\{ x \in \mathbb{R}^d \colon \langle a_i, x \rangle = 0, \; i = 1, \cdots, m \right\}$

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- $W := \{x \in \mathbb{R}^d \colon \langle a_i, x \rangle = 0, \ i = 1, \cdots, m\}$
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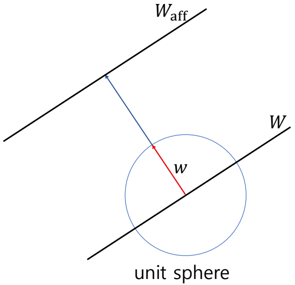
- $\tilde{W} := \operatorname{span} W_{\text{aff}}$
- $v := \Pi_W(z)$



- Write $x \in \tilde{W}$ as $x = hy$ for some $y \in W + w$, then
- $\langle z, x \rangle = \langle v, x \rangle + \langle z - v, x \rangle = h \langle v, y \rangle + h \langle z - v, w \rangle$

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- Thus, (CP) is equivalent to the problem

$$\begin{aligned} \text{maximize} \quad & \langle v, y \rangle \\ \text{subject to} \quad & \langle w, y \rangle = 1 \\ & y \in C \cap \tilde{W} \end{aligned}$$

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which can be, by the lemma, further reduced to

$$\begin{array}{rl} \text{maximize} & \langle v, y \rangle \\ \text{subject to} & \langle w, y \rangle = 1 \\ & y \in \Pi_L(C \cap \tilde{W}) \end{array} \tag{CP2D}$$

- Reformulate the problem:
  1. Uniformly randomly generate a subspace $\tilde{W}$ of codimension $m-1$

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- Given these, we are looking for the probabilities that the following optimization problem is (1) infeasible, (2) unbounded, (3) has a unique solution and that solution is in $\Pi_L(M \cap \tilde{W})$ some conic Borel set M:

$$\begin{aligned} \text{maximize} \quad & \langle v, y \rangle \\ \text{subject to} \quad & \langle w, y \rangle = 1 \\ & y \in \Pi_L(C \cap \tilde{W}) \end{aligned} \tag{CP2D}$$

Lemma 3

Let $\bar{C} \subseteq \mathbb{R}^2$ *be a closed convex cone. For a uniformly random* $\begin{bmatrix} v & w \end{bmatrix} \in O(2)$, *define*

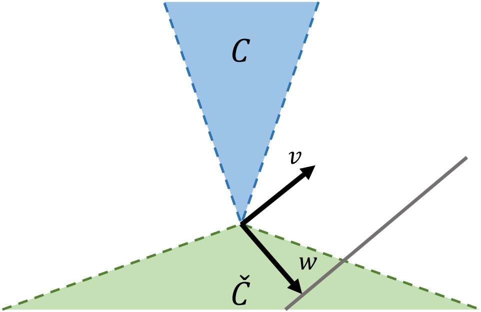
$$\bar{F} := \left\{ x \in \bar{C} \colon \langle w, x \rangle = 1 \right\},$$

then we have

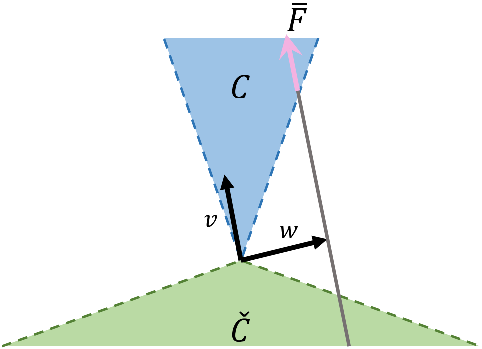
$$\begin{aligned} \Pr\left[\bar{F} = \emptyset\right] &= V_0(\bar{C}), \\ \Pr\left[\sup\left\{\langle v, x \rangle : x \in \bar{F}\right\} = \infty\right] &= V_2(\bar{C}), \\ \Pr\left[\operatorname{argmax}\left\{\langle v, x \rangle : x \in \bar{F}\right\} \in M\right] &= \Phi_1(\bar{C}, M) \end{aligned}$$

for any conic Borel set $M \subseteq \mathbb{R}^2$.

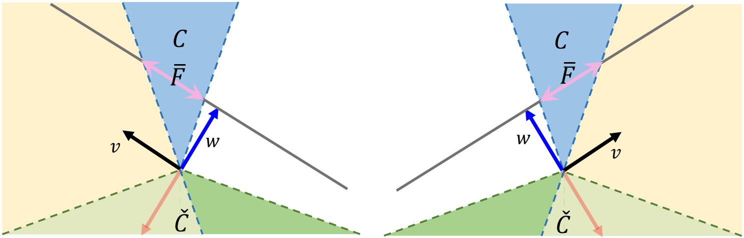
- $\Pr\left[\bar{F} = \emptyset\right] = V_0(\bar{C})$



- $\Pr\left[\sup\left\{\langle v, x\rangle : x \in \bar{F}\right\} = \infty\right] = V_2(\bar{C})$



- $\Pr\left[\operatorname{argmax}\left\{\langle v, x\rangle : x \in \bar{F}\right\} \in M\right] = \Phi_1(\bar{C}, M)$



- Left: argmax is on the left boundary edge
- Right: argmax is on the right boundary edge

- If M contains no boundary, we get $0 = \Phi_1(\bar{C}, M)$
- If M contains both boundaries, we get $\frac{1}{2} = V_1(\bar{C}) = \Phi_1(\bar{C}, M)$
- If M contains only one of the boundaries, we get $\frac{1}{4} = \frac{1}{2} V_1(\bar{C}) = \Phi_1(\bar{C}, M)$

Proof of Theorem 1.

By the previous discussions,

$$\Pr_{\substack{a_1,\cdots,a_m,z,\\ b_1,\cdots,b_m}}[\text{(CP) is infeasible}] = \Pr_{\tilde{W},L}\Pr_{v,w}[\text{(CP2D) is infeasible}]$$
$$= \mathbb{E}_{\tilde{W},L}\left[V_0(\Pi_L(C\cap\tilde{W}))\right]$$

and

$$\Pr_{\substack{a_1,\cdots,a_m,z,\\ b_1,\cdots,b_m}}[\text{(CP) is unbounded}] = \Pr_{\tilde{W},L}\Pr_{v,w}[\text{(CP2D) is unbounded}]$$
$$= \mathbb{E}_{\tilde{W},L}\left[V_2(\Pi_L(C\cap\tilde{W}))\right],$$

and similarly,

$$\Pr_{\substack{a_1,\cdots,a_m,z,\\ b_1,\cdots,b_m}}[\mathrm{sol}\,(\text{CP})\in M] = \Pr_{\tilde{W},L}\Pr_{v,w}\left[\mathrm{sol}\,(\text{CP2D})\in\Pi_L(M\cap\tilde{W})\right]$$
$$= \mathbb{E}_{\tilde{W},L}\left[\Phi_1(\Pi_L(C\cap\tilde{W}),\Pi_L(M\cap\tilde{W}))\right].$$

Theorem 4 (Kinematic formula)

Let $C \subseteq \mathbb{R}^d$ be a closed convex cone, and $W \subseteq \mathbb{R}^d$ be a uniformly random subspace of codimension $m \leq d-1$, and M be a conic Borel subset of $\mathbb{R}^d$. Then,

1. *(Random intersection formula)*

$$\mathbb{E}\left[\Phi_j(C \cap W, M \cap W)\right] = \Phi_{m+j}(C, M), \quad \textit{for } j = 1, \cdots, d-m,$$
$$\mathbb{E}\left[V_0(C \cap W)\right] = V_0(C) + V_1(C) + \cdots + V_m(C).$$

2. *(Random projection formula)*

$$\mathbb{E}\left[\Phi_j(\Pi_W(C), \Pi_W(M))\right] = \Phi_j(C, M), \quad \textit{for } j = 0, \cdots, d-m-1$$
$$\mathbb{E}\left[V_{d-m}(\Pi_W(C))\right] = V_{d-m}(C) + V_{d-m+1}(C) + \cdots + V_d(C).$$

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Proof of Theorem 1, cont'd.

$$\Pr_{\substack{a_1, \cdots, a_m, z, \\ b_1, \cdots, b_m}} [\text{(CP) is infeasible}] = \mathbb{E}_{\tilde{W}, L}\left[V_0(\Pi_L(C \cap \tilde{W}))\right]$$

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Proof of Theorem 1, cont'd.

$$\Pr_{\substack{a_1, \cdots, a_m, z, \\ b_1, \cdots, b_m}}[(\text{CP}) \text{ is unbounded}] = \mathbb{E}_{\tilde{W}, L}\left[V_2(\Pi_L(C \cap \tilde{W}))\right]$$

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$$= \mathbb{E}_{\tilde{W}}\left[V_2(C \cap \tilde{W}) + V_3(C \cap \tilde{W}) + \cdots + V_{d-m+1}(C \cap \tilde{W})\right]$$
$$= V_{m+1}(C) + V_{m+2}(C) + \cdots + V_d(C).$$

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$$\Pr_{\substack{a_1, \cdots, a_m, z, \\ b_1, \cdots, b_m}} \left[\mathrm{sol}\,(\mathrm{CP}) \in M\right] = \mathbb{E}_{\tilde{W}, L}\left[\Phi_1(\Pi_L(C \cap \tilde{W}), \Pi_L(M \cap \tilde{W}))\right]$$

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Proof of Theorem 1, cont'd.

$$\Pr_{\substack{a_1, \cdots, a_m, z, \\ b_1, \cdots, b_m}} [\mathrm{sol}\,(\mathrm{CP}) \in M] = \mathbb{E}_{\tilde{W}, L}\left[\Phi_1(\Pi_L(C \cap \tilde{W}), \Pi_L(M \cap \tilde{W}))\right]$$
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$$\Pr_{\substack{a_1, \cdots, a_m, z, \\ b_1, \cdots, b_m}} [\mathrm{sol}\,(\mathrm{CP}) \in M] = \mathbb{E}_{\tilde{W}, L}\left[\Phi_1(\Pi_L(C \cap \tilde{W}), \Pi_L(M \cap \tilde{W}))\right]$$
$$= \mathbb{E}_{\tilde{W}}\left[\Phi_1(C \cap \tilde{W}, M \cap \tilde{W})\right] = \Phi_m(C, M).$$

□

Any questions?