Point-Free Measure Theory

Junekey Jeon

UCSD

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The conventional, “pointed” measure theory

- Measure space $(X, \mathscr{A}, \mu)$
  - A *set* X
  - A *σ-algebra* $\mathscr{A}$ of subsets of X
  - A *measure* $\mu \colon \mathscr{A} \to [0, \infty]$

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- A *measurable function* $f \colon X \to \mathbb{R}$ is a function such that $f^{-1}\big[(-\infty, a)\big] \in \mathscr{A}$ for all $a \in \mathbb{R}$
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- Convergence theorems
  - Monotone convergence theorem
  - Fatou's lemma
  - Lebesgue's dominated convergence theorem
  - Vitali convergence theorem

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- Bizarre role of “points”

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- Pointwise a.e. convergence is not topological
  - $\nexists$ topology inducing pointwise a.e. convergence

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- $[0, \infty]$ can be replaced by any commutative monoid (e.g. locally convex space)
- We **don't** require μ to be countably-additive

Some facts about Boolean rings

Proposition 1

Let B be a Boolean ring.

1. *Any $a \in B$ is the additive inverse of itself: $a + a = 0$.*
2. *B is commutative.*
3. *If B does not have zero divisor, then $B \cong \mathbb{F}_2$.*
4. *$\operatorname{Spec} B = \operatorname{maxSpec} B$ and $B/\mathfrak{p} \cong \mathbb{F}_2$ for all $\mathfrak{p} \in \operatorname{Spec} B$.*

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    - (Closure under finite joins) $a, b \in \mathfrak{a}$ implies $a \vee b \in \mathfrak{a}$, and
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  - We call B *complete* if every subset of B admits the supremum (and the infimum)
    - $\wp(X)$ is complete
    - Borel/Lebesgue σ-algebras aren't

Topology on measured rings

- Given a measured ring (B, μ),
  - $B_{\mu<\infty} \coloneqq \{a \in B \colon \mu(a) < \infty\}$ is an ideal
  - For each $t \in B_{\mu<\infty}$, define

$$d_{\mu;t}(a, b) \coloneqq \mu(t(a + b))$$

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- e.g., if $\mu(a) = 0$ or $\mu(a) = \infty$ for all $a \in B$, then the topology is indiscrete

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- Suppose from now on every measure is semifinite

Topology on measured rings

Theorem 4

A semifinite measured ring (B, μ) is a complete Hausdorff topological ring if and only if:

1. *B is a complete Boolean ring, and*
2. *μ is strictly positive and order-continuous.*

- A measure μ is said to be *strictly positive*, if $\mu(a) = 0$ implies $a = 0$.
- A measure μ is said to be *order-continuous*, if for any increasing net $(a_\alpha)_{\alpha \in D}$ with the supremum $a = \bigvee_{\alpha \in D} a_\alpha$, we have

$$\mu(a) = \lim_{\alpha \in D} \mu(a_\alpha).$$

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Theorem 5

For a semifinite measured ring (B, μ), let $\overline{B}^{\mu}$ be the completion of B with respect to the topology induced by μ, then there uniquely exists a semifinite measure $\bar{\mu}\colon \overline{B}^{\mu} \to [0, \infty]$ such that

1. *$\bar{\mu}$ extends μ, and*
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Hence, $\overline{B}^{\mu}$ is a complete Boolean ring, and $\bar{\mu}$ is strictly positive and order-continuous.

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- We can do similar things with *locally convex space-valued measures* or *(possibly uncountable) collections of measures*

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Proposition 6

If μ is finite, then the Boolean ring $\mathscr{A} / \ker \mu$ is complete and $\bar{\mu}$ is order-continuous. Hence, $(\mathscr{A} / \ker \mu, \bar{\mu})$ is the Hausdorff completion of the measured ring $(\mathscr{A}, \mu)$.

- $\mathscr{A} / \ker \mu$ has the *countable chain condition (ccc)*
  - Every collection of nonzero disjoint elements must be countable

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  - L^p-spaces as completions with respect to the p-norm

Stone duality theorem

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Theorem 7 (Stone duality theorem)

The functor Spec *(with the Zariski topology) is a contravariant equivalence between the category of Boolean rings (with ring maps) and the category of Stone spaces (with continuous maps). The inverse functor is* Clopen *(or equivalently,* $\mathcal{C}(\,\cdot\,;\mathbb{F}_2)$*).*

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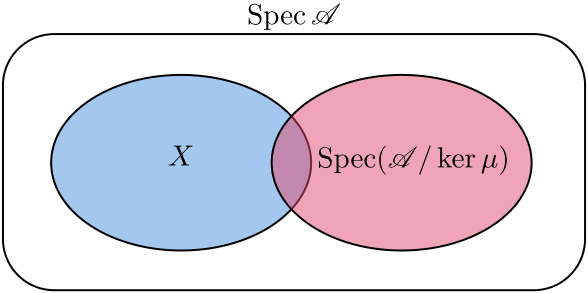
- (Uniform closure of clopen simple functions) $=$ (continuous functions on $\mathrm{Spec}\,B$)
- Therefore,

$$\mathcal{M}_b(B;\mathbb{R}) = \mathcal{C}(\mathrm{Spec}\,B;\mathbb{R})$$

Quotient by a.e. equivalence



$$\begin{array}{ccc}
\mathcal{M}_b(X;\mathbb{R}) & \xrightarrow{\ \cong\ } & \mathcal{C}(\operatorname{Spec}\mathscr{A};\mathbb{R}) \\
\text{a.e.}\downarrow & & \downarrow\text{restriction} \\
L^\infty(X,\mu;\mathbb{R}) & \xrightarrow{\ \cong\ } & \mathcal{C}(\operatorname{Spec}(\mathscr{A}/\ker\mu);\mathbb{R})
\end{array}$$

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- Reality in point-free measure theory:
  - measurable functions are “literally” continuous functions

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- Define
$$L^0(\mu; \mathbb{R}) := \varinjlim_U \mathcal{C}(U; \mathbb{R})$$
in the category of TVS's
  - Elements in $L^0(\mu; \mathbb{R})$ are *point-free measurable functions* on B

Unbounded measurable functions

$$L^0(\mu;\mathbb{R}) := \varinjlim_{U} \mathcal{C}(U;\mathbb{R})$$

- **Fact**: $L^0(X,\mu;\mathbb{R})$ is the completion of the space of simple functions with respect to the topology of convergence in measure

Unbounded measurable functions

Theorem 8

The topological vector space

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is precisely the completion of the space of real-valued clopen simple functions with respect to the topology of convergence in measure.

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- **Conjecture**: L^0 is the *sheafification* of L^∞

Some consequences

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- Convergence theorems for nets